1. At Stanford, happiness is measured in Stanjoys. While working on math homework without interruption, Lockensy earns Stanjoys at a continuous rate of $\frac{t^2}{7}$ Stanjoys per minute, where t is the number of minutes since she last resumed uninterrupted homework. While not doing her math homework, she earns Stanjoys at a constant rate of 1 per minute. During a 1-hour homework session, Lockensy takes an 18-minute phone call from her friend Chillsea. Given that Chillsea times her call to minimize Lockensy's total Stanjoys, compute the amount of Stanjoys that Lockensy will accumulate by the end of the hour session.

   Answer: 900

   Solution: Since $\frac{t^2}{7}$ is concave up, the longer Lockensy spends doing math without taking a break, the more happiness she earns on average. Thus, Chillsea should time her call perfectly in the middle of Lockensy's hour. This occurs if Chillsea calls at the 21 minute mark, allowing Lockensy to accumulate Stanjoys at two intervals of 21 minutes doing homework, and one interval of 18 minutes taking a call. The amount of happiness earned during each 21-minute interval is

$$\int_0^{21} \frac{t^2}{7} dt = \frac{t^3}{21}\Big|_{t=0}^{21}$$

$$= \frac{21^3}{21} = 441.$$

   The amount of happiness earned during the phone call is 18. Thus, the answer is $441 + 18 + 441 = \boxed{900}$ Stanjoys.

2. Let $f(x) = e^{\cos^2(x)}$ and $g(x) = e^{\sin^2(x)}$. Compute

$$\int_0^{\pi} f'(x)g'(x)\ \mathrm{d}x.$$

   Answer: $-\frac{\pi e}{2}$

   Solution: We first compute $f'(x)$ and $g'(x)$. Note that

$$f'(x) = \frac{d}{dx}e^{\cos^2(x)} = e^{\cos^2(x)}\frac{d}{dx}\cos^2(x) = e^{\cos^2(x)}2\cos(x)\frac{d}{dx}\cos(x) = e^{\cos^2(x)}(-2\cos(x)\sin(x))$$

   by two iterations of the Chain Rule, which states that $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. Similarly,

$$g'(x) = \frac{d}{dx}e^{\sin^2(x)} = e^{\sin^2(x)}\frac{d}{dx}\sin^2(x) = e^{\sin^2(x)}2\sin(x)\frac{d}{dx}\sin(x) = e^{\sin^2(x)}(2\cos(x)\sin(x))$$

   by two iterations of the Chain Rule. Recall the Pythagorean Identity, which states that $\cos^2(x) + \sin^2(x) = 1$ for any real x, and the double angle identities, which state that $\sin(2x) = 2\sin(x)\cos(x)$ and $\sin^2(2x) = \frac{1-\cos(4x)}{2}$. By the Pythagorean Identity and the double angle identities, we have

$$\begin{aligned}\int_0^\pi (-2\cos(x)\sin(x))(2\sin(x)\cos(x))e^{\cos^2(x)}e^{\sin^2(x)}\,dx &= \int_0^\pi -\sin^2(2x)e^{\cos^2(x)+\sin^2(x)}\,dx\\ &= -e\int_0^\pi \sin^2(2x)\,dx\\ &= -e\int_0^\pi \frac{1-\cos(4x)}{2}dx\\ &= -e\left[\frac{x}{2}-\frac{\sin(4x)}{8}\right]\Bigg|_0^\pi\\ &= \boxed{-\tfrac{\pi e}{2}}.\end{aligned}$$

3. Given real constants a, b, c, define the cubic polynomials

$$\begin{aligned}A(x) &= ax^3 + abx^2 - 4x - c\\ B(x) &= bx^3 + bcx^2 - 6x - a\\ C(x) &= cx^3 + cax^2 - 9x - b.\end{aligned}$$

Given that A, B, and C have local extrema at b, c, and a, respectively, compute abc. (A local extremum is either a local maximum or a local minimum.)

Answer: $\frac{6}{5}$

Solution: By the critical point theorem and differentiability of cubics everywhere, $A'(b) = B'(c) = C'(a) = 0$. That is,

$$A'(b) = 3ab^2 + 2ab^2 - 4 = 5ab^2 - 4 = 0,$$

$$B'(c) = 3bc^2 + 2bc^2 - 6 = 5bc^2 - 6 = 0,$$

and

$$C'(a) = 3ca^2 + 2ca^2 - 9 = 5ca^2 - 9 = 0.$$

Thus, $5ab^2 = 4, 5bc^2 = 6, 5ca^2 = 9$, implying

$$(5ab^2)(5bc^2)(5ca^2) = (5abc)^3 = 4(6)(9) = 6^3.$$

It follows that the answer is $\boxed{\frac{6}{5}}$.

Note: solving for a, b, c directly gives us that $a = c = \sqrt[3]{\frac{9}{5}}$ and $b = \sqrt[3]{\frac{8}{15}}$. This allows us to confirm that these critical points are local extrema via the second derivative test.

4. Let f be a continuous function satisfying $f(x^7 + 6x^5 + 3x^3 + 1) = 9x + 5$ for all reals x. Compute the integral

$$\int_{-9}^{11} f(x)\,\mathrm{d}x.$$

Answer: 100

Solution: Let

$$x = u^7 + 6u^5 + 3u^3 + 1.$$

Then

$$f(x) = f(u^7 + 6u^5 + 3u^3 + 1) = 9u + 5,$$

and

$$dx = \frac{d}{du}(u^7 + 6u^5 + 3u^3 + 1)\,du = (7u^6 + 30u^4 + 9u^2)\,du.$$

So the integral becomes:

$$\int_{x=-9}^{11} f(x)\,dx = \int_{u=-1}^{1} f(u^7 + 6u^5 + 3u^3 + 1)(7u^6 + 30u^4 + 9u^2)du$$
$$= \int_{u=-1}^{1} (9u+5)(7u^6 + 30u^4 + 9u^2)\,du.$$

Expanding the integrand gives

$$(9u+5)(7u^6 + 30u^4 + 9u^2) = 9u(7u^6 + 30u^4 + 9u^2) + 5(7u^6 + 30u^4 + 9u^2)$$
$$= 63u^7 + 270u^5 + 81u^3 + 35u^6 + 150u^4 + 45u^2.$$

Note that the integral of an odd function over a symmetric interval $[-a, a]$ is zero, so the integral becomes

$$\int_{-1}^{1} (35u^6 + 150u^4 + 45u^2)\,du = 5u^7 + 30u^5 + 15u^3 \Big|_{-1}^{1} = \boxed{100}.$$

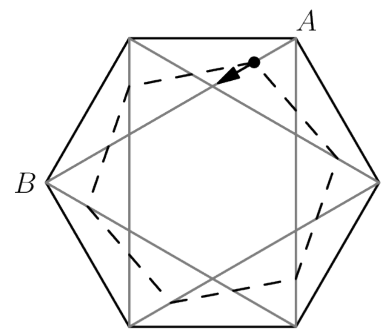
REMARK (Feel free to skip if too advanced). Construction of a continous function f. Let $g(x) = x^7 + 6x^5 + 3x^3 + 4x$. Note that g is injective since it is strictly increasing. It is a well-known theorem provable with point-set topology (Hausdorff and Compact spaces) that there exists an inverse $g^{-1}(x)$ of g which is continuous. Then, let $f(x) = 9g^{-1}(x) + 5$. Since $g^{-1}(x)$ is continuous, so is f. Moreover, $f(x^7 + 6x^5 + 3x^3 + 1) = f(g(x)) = 9g^{-1}(g(x)) + 4 = 9x + 5$ for all reals x, as desired.

5. Consider the set of all continuous and infinitely differentiable functions f with domain $[0, 2025]$ satisfying $f(0) = 0$, $f'(0) = 0$, $f'(2025) = 1$, and f'' being strictly increasing on the interval $[0, 2025]$. Compute the smallest real number M such that for all functions f in this set, $f(2025) < M$.

Answer: $\frac{2025}{2}$

Solution: We are looking for possible values for $f(2025)$. By the Fundamental Theorem of Calculus, $f(2025) = f(2025) - f(0) = \int_0^{2025} f'(x)dx$, which is just the area under f' on the interval $[0, 2025]$. Furthermore, since f'' is strictly increasing on the interval $[0, 2025]$, f' is convex on this interval. By property of convexity, f' must lie below the line between $(0,0)$ and $(2025,1)$ on this interval, so the area is no more than the area under the line. The area under the line is $\frac{1}{2}(2025)(1) = \frac{2025}{2}$, and so the answer is simply $M = \boxed{\frac{2025}{2}}$.

6. In the diagram, the larger regular hexagon has side length 1. A black dot moves with constant velocity from A to B, two vertices on the larger hexagon as shown in the diagram. The dashed regular hexagon is then drawn with one of its vertices on the black dot while remaining rotationally symmetric with respect to the center of the larger hexagon (i.e. the dashed hexagon has the same center as the large hexagon). Compute the average area of the dashed hexagon over time as the black dot travels from A to B.



Answer: $\frac{3\sqrt{3}}{4}$

Solution: Note that the only relevant information is the distance from the black dot to the center of the hexagon. Letting this be s, the area of the dashed hexagon is therefore $\frac{3\sqrt{3}s^2}{2}$. Let C denote the foot of the altitude from the center of the hexagon to the grey diagonal that the black dot moves along, and let the distance from the black dot to C be t. By the Pythagorean theorem, $s^2 = \frac{1}{4} + t^2$, so we can express the area as a function of t as

$$A(t) = \frac{3\sqrt{3}}{2}\left(\frac{1}{4} + t^2\right).$$

Moreover, notice that as the black dot travels from A to C, the value of t varies from $\frac{\sqrt{3}}{2}$ to 0. Observe that we can ignore the second half of the path by symmetry. Hence, the desired integral is

$$\frac{1}{\frac{\sqrt{3}}{2}}\int_0^{\frac{\sqrt{3}}{2}} A(t)\,dt = \frac{3\sqrt{3}}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\cdot\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{1}{4}+t^2\right)dt = \boxed{\frac{3\sqrt{3}}{4}}.$$

7. Compute

$$\lim_{t\to 0}\left(\prod_{n=2}^{\infty}\left(1+\frac{t}{n^2+n}\right)\right)^{\frac{1}{t}}.$$

Answer: $e^{\frac{1}{2}}$

Solution: SOLUTION ONE: Let $r = \frac{1}{t}$. Then, the limit becomes

$$\lim_{r\to\infty}\left(\prod_{n=2}^{\infty}\left(1+\frac{\frac{1}{n(n+1)}}{r}\right)\right)^{r}.$$

Note that, for each n,

$$\lim_{r\to\infty}\left(1+\frac{\frac{1}{n(n+1)}}{r}\right)^{r} = e^{\frac{1}{n(n+1)}}$$

by the limit definition of e^x, with $x = \frac{1}{n(n+1)}$. Thus,

$$\begin{aligned}&\lim_{r\to\infty}\left(\prod_{n=2}^{\infty}\left(1+\frac{\frac{1}{n(n+1)}}{r}\right)\right)^r\\&=\prod_{n=2}^{\infty}\left(\lim_{r\to\infty}\left(1+\frac{\frac{1}{n(n+1)}}{r}\right)^r\right)\\&=\prod_{n=2}^{\infty}e^{\frac{1}{n(n+1)}}\\&=e^{\sum_{n=2}^{\infty}\frac{1}{n(n+1)}}\\&=\boxed{e^{\frac{1}{2}}}\end{aligned}$$

by telescoping series $\sum_{n=2}^{\infty}\frac{1}{n(n+1)} = \sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}$.
Note that we are allowed to move the limit inside the product as each limit pointwise converges and strictly increases from 1, so the overall limit of the product does approach $\boxed{e^{\frac{1}{2}}}$.

SOLUTION TWO: Let $L(t) = \left(\prod_{n=2}^{\infty}\left(1+\frac{t}{n^2+n}\right)\right)^{\frac{1}{t}} = \left(\prod_{n=2}^{\infty}\left(\frac{n^2+n+t}{n^2+n}\right)\right)^{\frac{1}{t}}$. We wish to compute $\lim_{t\to 0} L(t)$. To do so, we first take natural log on $L(t)$ to convert the product of the sequence into a summation series.

$$\begin{aligned}&\ln(L(t))\\&=\ln\left(\left(\prod_{n=2}^{\infty}\left(\frac{n^2+n+t}{n^2+n}\right)\right)^{\frac{1}{t}}\right)\\&=\frac{1}{t}\sum_{n=2}^{\infty}(\ln(n^2+n+t)-\ln(n^2+n))\\&=\sum_{n=2}^{\infty}\left(\frac{\ln(n^2+n+t)-\ln(n^2+n)}{t}\right).\end{aligned}$$

Note that, as t approaches 0, the inside expression $\frac{\ln(n^2+n+t)-\ln(n^2+n)}{t}$ is just the definition of derivative of $\ln(x)$ at $x = n^2+n$. Thus, $\lim_{t\to 0}\left(\frac{\ln(n^2+n+t)-\ln(n^2+n)}{t}\right) = \frac{1}{n^2+n}$. Therefore,

$$\begin{aligned}
&\ln\Big(\lim_{t\to 0} L(t)\Big) \\
&= \lim_{t\to 0} \ln(L(t)) \\
&= \sum_{n=2}^{\infty} \lim_{t\to 0}\left(\frac{\ln(n^2+n+t)-\ln(n^2+n)}{t}\right) \\
&= \sum_{n=2}^{\infty} \frac{1}{n^2+n} \\
&= \sum_{n=2}^{\infty} \frac{1}{n(n+1)} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Finally, $\lim_{t\to 0} L(t) = e^{\ln(\lim_{t\to 0} L(t))} = \boxed{e^{\frac{1}{2}}}$.

8. Let R be the region in the complex plane enclosed by the curve $f(\theta) = e^{i\theta} + e^{2i\theta} + \frac{1}{3}e^{3i\theta}$ for $0 \leq \theta \leq 2\pi$. Compute the perimeter of R.

Answer: $2\sqrt{3} + \frac{8\pi}{3}$

Solution: We may write $f(\theta) = \frac{1}{3}\big((1+e^{i\theta})^3 - 1\big)$ and notice that the perimeter of this curve encloses 0. Let us first let θ^* be the first $\theta > 0$ such that $f(\theta^*)$ is a real number (and thus on the negative real line). Then, if the perimeter of the figure from 0 to θ^* is P, our answer is $2P$.

To compute θ^*, we want solutions to $(1+e^{i\theta})^3 = 3r+1$ for some real r. Equivalently, $e^{i\theta} = -1 + \omega \cdot \sqrt[3]{3r+1}$ for ω a third root of unity and thus this latter quantity must have unit norm. Letting $s = \sqrt[3]{3r+1}$ (which is still a real number), we want

$$1 = (-1+s\omega)(-1+s\omega^2) = 1 + s^2 - s(\omega+\omega^2) = 1 + s^2 + s \Longrightarrow s = 0, -1$$

Here we used that $\overline{\omega} = \omega^2$ and for any complex number, $z\overline{z} = |\, z\,|^2$. The first intersection therefore occurs at $s = -1$. Thus, $e^{i\theta} = -1 - \omega$. Here, $\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, so $\theta^* = \frac{2\pi}{3}$.

Now, let us compute P. We can express this as a Riemann sum over θ, where we are tasked with finding the distance from $f(\theta)$ to $f(\theta + \mathrm{d}\theta)$. By Taylor Series, we can write $f(\theta + \mathrm{d}\theta) \sim f(\theta) + f'(\theta) \cdot \mathrm{d}\theta$ (complex derivatives are taken component-wise) and thus this distance is $|\, f'(\theta)| \cdot \mathrm{d}\theta$.

Noting that $f'(\theta) = ie^{i\theta} + 2ie^{2i\theta} + ie^{3i\theta}$, we may calculate

$$\begin{aligned}
|\, f'(\theta)|^2 &= (\sin(\theta) + 2\sin(2\theta) + \sin(3\theta)^2 + (\cos(\theta) + 2\cos(2\theta) + \cos(3\theta)^2 \\
&= 6 + 4(\sin\theta\sin(2\theta) + \cos\theta\cos(2\theta)) + 2(\sin\theta\sin(3\theta) + \cos\theta\cos(3\theta)) + 4(\sin(2\theta)\sin(3\theta) + \cos(2\theta)\cos(3\theta)) \\
&= 6 + 8\cos(\theta) + 2\cos(2\theta) \\
&= 4 + 8\cos(\theta) + 4\cos^2(\theta) \\
&= (2 + 2\cos(\theta))^2
\end{aligned}$$

via the cosine sum identity. Hence, it follows that

$$P = \int_0^{2\pi/3} 2 + 2\cos\theta \, \mathrm{d}\theta = \frac{4\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) = \sqrt{3} + \frac{4\pi}{3}.$$

Doubling this gives the answer of $\boxed{2\sqrt{3} + \frac{8\pi}{3}}$.

9. Consider the function

$$f(x) = \frac{(2025 + x)\ln(2025 + x)}{x^3 - 6x^2 + 11x - 6}$$

defined for all real numbers except $x = 1, 2$, and 3. For a positive integer n, let d_n denote $\frac{f^{(n)}(0)}{n!}$, where $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$. Compute $\lim_{n\to\infty} d_n$.

Answer: $-1013\ln(2026)$

Solution: Note that $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$. Define the function

$$g(x) = \frac{-(2025 + x)\ln(2025 + x)}{(x - 2)(x - 3)}$$

so that $f(x) = \frac{g(x)}{1-x}$. Since $g(x)$ is analytic at 0, it has a Taylor expansion:

$$g(x) = \sum_{j=0}^{\infty} g_j x^j.$$

Also, note that:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

As $f(x)$ is also analytic at 0,

$$f(x) = g(x) \cdot \frac{1}{1-x} = \left(\sum_{j=0}^{\infty} g_j x^j\right)\left(\sum_{k=0}^{\infty} x^k\right).$$

Furthermore, note that d_n is the coefficient of x^n in the Taylor series expansion of $f(x)$. Thus, we can compute it to be:

$$d_n = \sum_{j+k=n} g_j \cdot 1 = \sum_{j=0}^{n} g_j.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} d_n = \sum_{j=0}^{\infty} g_j = g(1).$$

We can do this as $g(x)$ has radius of convergence 2, so the Taylor series does converge at $x = 1$.

Finally, we compute $g(1)$. Substituting $x = 1$ into $g(x)$,

$$g(1) = \frac{-(2025+1)\ln(2025+1)}{(1-2)(1-3)} = \boxed{-1013\ln(2026)}.$$

10. Let S be a sphere centered at the origin O with radius 1. Let $A = (0,0,1)$. Let B and C be two points (not equal to A or each other) on the sphere's surface with equal z coordinate. Let minor arc AB be the shorter distance on the intersection of the sphere with the plane containing A, B, and O and similarly, let minor arc AC be the shorter distance on the intersection of the sphere with the plane containing A, C, and O. Let the minor arc BC be the shorter distance on the intersection of the sphere with the plane parallel to the xy plane and at the same z value as B (or either if the two arcs have the same length). The three arcs combined divide the surface of the sphere into two regions. Given that the area of smaller region enclosed by arcs AB, AC, and BC is 3, compute the minimum possible sum of the arc lengths of AB, AC, and BC.

Answer: $\frac{4\pi}{3} + \sqrt{3}$

Solution: The difficult part is to derive the following formulas for surface area and length of a triangle on a sphere. Note that the two equilateral sides flow along points of longitude which go down of length θ, the polar angle. The last side is therefore the azimuthal angle, with arc length ϕ. Arc length of the triangle is rather easy to see, since 2 sides are just $S = r\theta = \theta$. However, the last side is slightly nontrivial, namely because your radius is bases on the radius in the x-y plane, but using spherical geometry (or, simply some right-triangle geometry in 3D space), we get $S = r\phi = \phi\sin(\theta)$.

To calculate the surface area, we can think of encompasses area as a surface of revolution of the sphere around the y-axis, but not a full revolution. To this, we take $x = \sqrt{1-y^2}$ as your function you are revolving. We have:

$$\frac{dx}{dy} = \frac{-y}{\sqrt{1-y^2}}$$

To get:

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \frac{1}{\sqrt{1-y^2}} = \frac{1}{x}$$

Now, note that we are not doing a full 2π revolution, but only for ϕ radians. Thus, the formula for the surface area is:

$$\int_{-1}^{x} \phi x' \frac{1}{x'}\, dx' = \int_{-1}^{x} \phi\, dx' = \phi(x+1) = \phi(1-\cos\theta)$$

With $x = -\cos\theta$ when reparametrizing, since we take $(-1,0)$ to be the point where $\theta = 0$ on the sphere, extending continuously to $(1,0)$ to be the point where $\theta = \pi$.

Surface area is more nontrivial, but this can be seen using proportionality between how it varies based on the polar and azimuthal angle. Note that based on the positioning, the surface area is proportional to ϕ. For the polar angle proportionality, note that the height changes at a differential amount by $dz = -\sin(\theta)\ d\theta$, so integrating both sides from $\theta' \in [0, \theta]$ yields the surface area proportionality of $1 - \cos(\theta)$. Putting this together, one can determine the proportionality constant by plugging in values for $\phi = 2\pi$, and $\theta = \frac{\pi}{2}$ for the surface area of the hemisphere to get the proportionality constant of 1. It follows that:

$$\phi(1 - \cos(\theta)) = \text{Surface Area} = 3$$

$$2\theta + \phi\sin(\theta) = \text{Arc Length} = L$$

So to minimize the arc length of the 3 sides, this becomes a standard optimization problem. In particular:

$$\phi = \frac{3}{1 - \cos(\theta)} \Rightarrow 2\theta + \frac{3\sin(\theta)}{1 - \cos(\theta)} = L$$

So by finding extrema:

$$\begin{aligned}\frac{dL}{d\theta} &= 2 + 3\frac{\cos(\theta)(1 - \cos(\theta)) - \sin(\theta)(\sin(\theta))}{(1 - \cos(\theta))^2} \\ &= 2 - \frac{3}{1 - \cos(\theta)} \\ &= 0\end{aligned}$$

Which implies:

$$\frac{3}{2} = 1 - \cos(\theta)$$

Or:

$$\cos(\theta) = -\frac{1}{2}$$

So the polar angle is:

$$\theta = \frac{2\pi}{3}$$

Now, there are more solutions, but $\theta \in [0, \pi]$, so we ignore the others. Indeed, one can verify at $\theta = 0$ and $\theta = \pi$, we have $L(0) = 0, L(\pi) = 2\pi$, and $\boxed{L\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3} + \sqrt{3}}$ but at $L(0)$, the surface area is forced to have area 0 (this makes sense).

TB. *This is an estimation question used for tiebreaking purposes. Ties on this test will be broken by absolute distance from the correct answer to this question.*

Let

$$I = \int_{\pi}^{2025\pi} \frac{\sin^2(x)}{\ln x}\ \mathrm{d}x.$$

Estimate I in the decimal form $abcdef.ghij$, where $a, b, c, d, e, f, g, h, i, j$ are decimal digits each between 0 and 9, inclusive (leading zeros are possible).

Answer: 419.7566639598717

Solution: