1. Carolina has a jar of red, green, and white jelly beans. She replaces 400 red beans with green ones, then 200 green beans with white ones. Initially, the percentage of red beans minus the percentage of green beans was $X\%$. After the replacements, this difference became $(X - 2.5)\%$. Compute the number of beans in the jar.

   Answer: $24,000$

   Solution: Overall, the number of red jelly beans decreased by 400. The number of green jelly beans first increased by 400 after the first operation and then decreased by 200 by the second operation, leaving an overall 200 increase. Therefore, the overall change in the difference of red and green jelly beans is 600 jelly beans. Since this is 2.5% of the entire jar, there are

   $$\frac{100}{2.5} \cdot 600 = \boxed{24,000}$$

   total jelly beans in the jar.

   REMARK: One can let r, g, and w be the number of red, green, and white jelly beans respectively. Then after the first operation we have $r - 400, g + 400$, and w red, green, and white jelly beans respectively. After the second operation, we have $r - 400, g + 200$, and $w + 200$ red, green, and white jelly beans respectively. Then since $(r - 400) - (g + 200) = -600$, this uses algebra to formalize our argument above.

2. Six friends—Andrea, Blake, Camila, Dean, Ethan, and Francis—drink a combined total of 32 cups of boba. Each person drinks at least one cup, no two friends drink the same number of cups, and each cup is fully finished by a single person. One friend, known as the "Boba Champion", drinks as many cups as all the others combined. Compute the product of the number of cups of boba consumed by the five friends who aren't the Boba Champion.

   Answer: 144

   Solution: Let a, b, c, d, e, f be the number of cups of boba each of the six friends drank. Without loss of generality, assume that f corresponds to the number of cups the Boba Champion drank. Then $a + b + c + d + e = f$ and $a + b + c + d + e + f = 2f = 32$ so $f = 16 = a + b + c + d + e$. Without loss of generality, assume that $a < b < c < d < e$. Since a, b, c, d, e are distinct positive integers, we get that their minimal sum is $1 + 2 + 3 + 4 + 5 = 15$. This is one less than our desired 16. So we need to increase one of the numbers by 1, while keeping the numbers distinct. The only possibility is to increase e to 6. Therefore, the only possible value for (a, b, c, d, e) is $(1, 2, 3, 4, 6)$. The answer is thus $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 = \boxed{144}$.

3. The absolute values of all three roots of the polynomial $x^3 - 147x + c$ are primes. Compute $|c|$. (A prime number is a positive integer greater than 1 whose only positive divisors are 1 and itself.)

   Answer: 286

   Solution: Let p, q, and r be the roots of the polynomial. Without loss of generality, assume $\mid p \mid \leq \mid q \mid \leq \mid r \mid$. By Vieta's formulas, $p + q + r = 0 \Longrightarrow q + r = -p$. Note that not all of $\mid p \mid, \mid q \mid, \mid r \mid$ can be odd. If they were, p, q, r would be odd, and $q + r = -p$ would be even. Therefore, $\mid -p \mid = \mid p \mid$ would be even, contradicting the assumption that $\mid p \mid$ is odd. Thus at least one of $|\ p\ |, |\ q\ |, |\ r\ |$ is even. Since $|\ p\ |, |\ q\ |, |\ r\ |$ are prime and $|\ p\ |$ is the smallest, this implies $|\ p\ | = 2$. Applying Vieta's formulas again, we find that

$$pq + qr + pr = -147 \implies$$

$$p(q + r) + qr = p(-p) + qr = qr - 4 = -147.$$

Note that here we relied on $p(-p) = -4$ whenever $|\ p\ | = 2$. Then, $qr = -143 = -11 \cdot 13$. From this prime factorization, we find $(q, r) = (-11, 13)$ or $(11, -13)$. In either case, by another application of Vieta's formulas, $|\ c\ | = |-pqr\ | = 2 \cdot 11 \cdot 13 = \boxed{286}$.

4. Compute the sum of the real roots of $(x^2 + 16x + 48)(x^2 + 24x + 128) = 44^{2^{2025}}$. (Roots are counted with multiplicity.)

   Answer: -20

   Solution: Factor the quadratic expressions and observe that we have $(x + 4)(x + 12)(x + 8)(x + 16) = 44^{2^{2025}}$. Expand the factorized pairs $(x + 4)(x + 16) = x^2 + 20x + 64$ and $(x + 8)(x + 12) = x^2 + 20x + 96$. Noticing the symmetry, substitute $y = x^2 + 20x + 80$. Then, we have

   $$(y - 16)(y + 16) = 44^{2^{2025}} \Rightarrow y^2 - 256 = 44^{2^{2025}} \Rightarrow y^2 = 256 + 44^{2^{2025}}.$$

   We define $c = \sqrt{256 + 44^{2^{2025}}}$, so that $y = \pm c$. For our purposes, it suffices to note that c is a very large number, which we can formally express as $c > 1000$. Now, we have two ways to finish:

   SOLUTION ONE. Solving for x, we obtain either $x^2 + 20x + 80 - c = 0$ or $x^2 + 20x + 80 + c = 0$. Since $20^2 - 4(80 + c) < 400 - 4(1000) < 0$, applying the quadratic formula shows that $x^2 + 20x + 80 + c = 0$ has no real solutions. On the other hand, by the same quadratic formula argument, since $400 - 4(80 - c) = 80 + 4c > 4000 > 0$, it follows that $x^2 + 20x + 80 - c = 0$ does have real solutions (this method is known as checking the discriminant of a quadratic). By Vieta's formulas on this quadratic, the sum of the real roots is $\boxed{-20}$.

   SOLUTION TWO. Instead of applying the quadratic formula, we may complete the square and observe that $x^2 + 20x + 80 = (x + 10)^2 - 20 \geq -20$ for real x. Since c is a large number, $x^2 + 20x + 80$ cannot be $-c$. However, since $c > -20$, it follows that $x^2 + 20x + 80 = c$ has two real roots. This provides an alternative way to determine that the real roots satisfy $x^2 + 20x + 80 = c$, from which Vieta's formulas gives an answer of $\boxed{-20}$.

   SOLUTION THREE. This is a separate solution from the first two. Recall that the equation can be rewritten as $(x + 4)(x + 12)(x + 8)(x + 16) = 44^{2^{2025}}$. Now, let $y = x + 10$. Rewriting the equation in terms of y, we obtain $(y - 6)(y + 2)(y - 2)(y + 6) = (y - 2)(y + 2)(y - 6)(y + 6) = (y^2 - 4)(y^2 - 36) = 44^{2^{2025}}$. Let $z = y^2$ so that we have

   $$(z - 4)(z - 36) - 44^{2^{2025}} = z^2 - 40z + 144 - 44^{2^{2025}} = 0.$$

   By the quadratic formula, $z = \frac{40 \pm \sqrt{(-40)^2 + 4(44^{2^{2025}} - 144)}}{2}$. Since $z = y^2 \geq 0$ whenever x, and hence y, is real, we only consider the positive value of z. For this positive value, the two possible values y are of form y_1, y_2 with $y_1 = -y_2$, since they square to the same value. Then if x_1, x_2 are the real roots corresponding to y_1, y_2 respectively, $(x_1+x_2)+20=(x_1+10)+(x_2+10)=y_1+y_2=0$ and $x_1+x_2=\boxed{-20}$.

5. Given real numbers s, m, and t such that $s^2+m^2+t^2=8$, $s^3+m^3+t^3=11$, and $s^4+m^4+t^4=25$, compute the value of $(s+m+t)(s+m-t)(s-m+t)(s-m-t)$.

Answer: -14

Solution: We use the identity $s^4+m^4+t^4=(s^2+m^2+t^2)^2-2(s^2m^2+s^2t^2+m^2t^2)$ to express $s^2m^2+s^2t^2+m^2t^2$ as $\frac{(s^2+m^2+t^2)^2}{2}-\frac{s^4+m^4+t^4}{2}$. Therefore:

$$\begin{aligned}
&(s+m+t)(s+m-t)(s-m+t)(s-m-t)\\
&=s^4+m^4+t^4-2(s^2m^2+s^2t^2+m^2t^2)\\
&=s^4+m^4+t^4-2\left[\frac{(s^2+m^2+t^2)^2}{2}-\frac{s^4+m^4+t^4}{2}\right]\\
&=2(s^4+m^4+t^4)-(s^2+m^2+t^2)^2\\
&=2(25)-8^2\\
&=50-64\\
&=\boxed{-14}.
\end{aligned}$$

REMARK. Alternatively, one can directly substitute $2(s^2m^2+s^2t^2+m^2t^2)$ as $(s^2+m^2+t^2)^2-(s^4+m^4+t^4)$ above without first solving for $s^2m^2+s^2t^2+m^2t^2$.

6. Compute the number of integers between 1 and 2025 inclusive that *cannot* be represented as $x(\lceil x\rceil+\lfloor x\rfloor)$ for any positive real number x.

Answer: 1002

Solution: Let $f(x)=x(\lceil x\rceil+\lfloor x\rfloor)$. We will instead compute the number of integers that are expressible as $f(x)$ for some x.

Now, let us partition the space of all positive real numbers by their floor: in particular, suppose that $\lfloor x\rfloor=n$. If $x=n$, then $f(x)=2n^2$ (as $\lceil n\rceil=n$).

Else, if $n<x<n+1$, we claim that the integers which can be expressed as $f(x)$ are exactly the integers from $2n^2+n+1$ through $2n^2+3n$, inclusive. Indeed, in this interval we have that $f(x)=x(2n+1)$. Evaluating at $x=n$ and $n+1$ gives $2n^2+n$ and $2n^2+3n+1$, respectively. Since $f(x)$ is strictly increasing and continuous on this range, it follows that the expressible integers for $x\in(n,n+1)$ (i.e., excluding the endpoints) are exactly $[2n^2+n+1, 2n^2+3n]$.

Therefore, we have two types of expressible integers: those of the above form and those of the form $2n^2$. Note that these two types of ranges never intersect, as

$$2n^2+3n<2(n+1)^2=2n^2+4n+2<2(n+1)^2+(n+1)+1.$$

The largest n we have to consider is 31, as $2\cdot 32^2=2048>2025$ and $2\cdot 31^2+3\cdot 31=2015\le 2025$. For each n, the total number of integers expressible by its ranges is $1+((2n^2+3n)-(2n^2+n+1)+1)=2n+1$. Therefore, the total number of expressible integers is

$$\sum_{n=1}^{31} 2n+1 = 31 + 31 \cdot 32 = 1023$$

and our final answer is $2025 - 1023 = \boxed{1002}$.

7. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all positive integers $n \geq 3$. Let $S = \{2^{F_2}, 2^{F_3}, ..., 2^{F_{27}}, 2^{F_{28}}\} = \{2, 4, ..., 2^{196418}, 2^{317811}\}$ be a set. If

$$P = \prod_{x \in S} \prod_{y \in S} x^{\log_3 y}$$

is the product of $x^{\log_3 y}$ across all ordered pairs of elements (x, y) in S, including when $x = y$, compute $\sqrt{\log_2 P}$.

Answer: $832038\sqrt{\log_3 2}$

Solution: We first show that for any set

$$S = \{2^{a_1}, 2^{a_2}, ..., 2^{a_n}\}$$

(where $a_1 \neq a_2 \neq \cdots \neq a_n$), we have

$$\log_2 P = (a_1 + a_2 + \cdots + a_n)^2 \log_3 2.$$

Indeed, we have

$$\log_2 P = \sum_{(x,y) \in S} \log_2(x^{\log_3 y}) = \sum_{(x,y) \in S} \log_2 x \cdot \log_3 y.$$

Letting $x = 2^{a_i}$ and $y = 2^{a_j}$, this becomes

$$\sum_{1 \leq i,j \leq n} \log_2(2^{a_i}) \cdot \log_3(2^{a_j}) = \sum_{1 \leq i,j \leq n} a_i a_j \log_3 2 = (a_1 + a_2 + \cdots + a_n)^2 \log_3 2,$$

as desired.

In the case of our problem, we have

$$S = \{2^{F_2}, 2^{F_3}, ..., 2^{F_{27}}, 2^{F_{28}}\},$$

so

$$\log_2 P = (F_2 + F_3 + \cdots + F_{28})^2 \log_3 2.$$

To compute $F_2 + F_3 + \cdots + F_{28}$, we will prove the identity

$$F_1 + F_2 + \cdots + F_m = F_{m+2} - 1$$

by induction on m.

Base case: $m = 1$. Then $F_1 = 1$ and $F_3 - 1 = 2 - 1 = 1$, so the identity holds.

Inductive step: Assume the identity holds for $m = k$, i.e.

$$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1.$$

We want to show it holds for $m = k + 1$, that is,

$$F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1.$$

Using the inductive hypothesis,

$$F_1 + \cdots + F_k + F_{k+1} = (F_{k+2} - 1) + F_{k+1} = F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1.$$

Thus, by induction, the identity holds for all $m \geq 1$.

Applying the identity with $m = 28$ gives

$$F_1 + F_2 + \cdots + F_{28} = F_{30} - 1,$$

so

$$F_2 + F_3 + \cdots + F_{28} = (F_{30} - 1) - F_1 = F_{30} - 2.$$

Therefore,

$$\log_2 P = (F_{30} - 2)^2 \log_3 2 \quad \text{and} \quad \sqrt{\log_2 P} = (F_{30} - 2)\sqrt{\log_3 2}.$$

We are given $F_{27} = 196418$ and $F_{28} = 317811$. Then

$$F_{29} = F_{28} + F_{27} = 514229, \quad F_{30} = F_{29} + F_{28} = 832040.$$

Thus, the final answer is

$$\boxed{832038\sqrt{\log_3 2}}.$$

8. Let Z be the set of all complex numbers z such that $|z| = 1$ and

$$z^8 + iz^6 - (1+i)z^5 + (1-i)z^3 + iz^2 - 1 = 0,$$

with the polar angle θ_z (the angle in $[0, 2\pi)$ that z makes with the positive real axis, measured counterclockwise) lying in the interval $[0, \pi)$. Compute the sum of all distinct values of θ_z for z in Z.

Answer: $\frac{29\pi}{12}$ **or** 435°

Solution: SOLUTION ONE: Let $p(z) = z^8 + iz^6 - (1+i)z^5 + (1-i)z^3 + iz^2 - 1$. Through some trial and error (one method being first factoring out $z^2 + i$ from each of $z^8 + iz^6 = (z^2 + i)z^6$, $-(1+i)z^5 + (1-i)z^3 = (z^2 + i)(-(1+i)z^3)$, and $iz^2 - 1 = (z^2 + i)i$ before factoring $z^6 - (1+i)z^3 + i$ similarly), we determine that $p(z)$ factors as

$$p(z) = (z^2 + i)(z^6 - (1+i)z^3 + i) = (z^2 + i)(z^3 - i)(z^3 - 1).$$

From $z^2 + i = 0$, we get $z^2 = -i = e^{\frac{3\pi}{2}}$, implying $z = \pm e^{\frac{3\pi}{4}}$. These have polar angles $\pm\frac{3\pi}{4}$, giving one solution $\theta = \frac{3\pi}{4}$ in the desired range $[0, \pi)$. Similarly, from $z^3 - i = 0$ we get $z^3 = i = e^{\frac{\pi}{2}} = e^{\frac{5\pi}{2}}$, giving solutions $\theta = \frac{\pi}{6}, \frac{5\pi}{6} = \frac{\frac{\pi}{2}}{3}, \frac{\frac{5\pi}{2}}{3}$. Finally, from $z^3 - 1 = 0$, we get $z^3 = 1$, from which the first and second roots of unity (i.e., $e^0, e^{\frac{2\pi}{3}}$) give solutions $\theta = 0, \frac{2\pi}{3}$. Since $\mid z \mid = 1$ in all cases, the answer is

$$\frac{3\pi}{4}+\frac{\pi}{6}+\frac{5\pi}{6}+0+\frac{2\pi}{3}=\boxed{\frac{29\pi}{12}}.$$

SOLUTION TWO: Dividing the given equation by iz^4 and rearranging, we arrive at

$$\frac{1}{i}\left(z^4-\frac{1}{z^4}\right)+\left(z^2+\frac{1}{z^2}\right)=\frac{1}{i}\left(z-\frac{1}{z}\right)+z+\frac{1}{z}.$$

Let $z=e^{i\theta}$ (this captures the information $|z|=1$). Recall that

$$z^n+\frac{1}{z^n}=e^{in\theta}+e^{-in\theta}=2\cos n\theta$$

and

$$\frac{1}{i}\left(z^n-\frac{1}{z^n}\right)=\frac{1}{i}\left(e^{in\theta}-e^{-in\theta}\right)=2\sin n\theta$$

for a positive integer n. In particular, for $n=4,2,1$ we have:

$$\frac{1}{i}\left(z^4-\frac{1}{z^4}\right)=2\sin 4\theta,\quad z^2+\frac{1}{z^2}=2\cos 2\theta,$$

$$\frac{1}{i}\left(z-\frac{1}{z}\right)=2\sin\theta,\quad z+\frac{1}{z}=2\cos\theta.$$

Substituting these into the equation gives:

$$2\sin 4\theta+2\cos 2\theta=2\sin\theta+2\cos\theta.$$

Dividing both sides by 2, we obtain

$$\sin 4\theta+\cos 2\theta=\sin\theta+\cos\theta$$

or

$$\cos 2\theta-\cos\theta=\sin\theta-\sin 4\theta.$$

Applying the sum to product identities,

$$-2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)=2\sin\left(-\frac{3\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right).$$

Then $\sin\frac{3\theta}{2}=0$ (noting that $\sin\left(-\frac{3\theta}{2}\right)=-\sin\left(\frac{3\theta}{2}\right)$) or $\sin\frac{\theta}{2}=\cos\frac{5\theta}{2}$. Looking at the unit circle, the first equation implies $\frac{3\theta}{2}$ is a multiple of π, giving solutions $\theta=0$ and $\theta=\frac{2\pi}{3}$ in the desired range $[0,\pi)$. The second equation can be rewritten as

$$\sin\frac{\theta}{2}=\sin\left(\frac{\pi}{2}-\frac{5\theta}{2}\right),$$

which yields (when similarly looking at the unit circle) the solutions $\theta=\frac{3\pi}{4}$ and $\theta=\frac{\pi}{6},\frac{5\pi}{6}$. Therefore, the answer is again

$$0+\frac{2\pi}{3}+\frac{3\pi}{4}+\frac{\pi}{6}+\frac{5\pi}{6}=\boxed{\frac{29\pi}{12}}.$$

9. Define a sequence $\{a_n\}$ by $a_1=5$ and $a_{n+1}=\frac{5a_n+\sqrt{21a_n^2+4}}{2}$ for $n\geq 1$. Compute the value of

$$\sum_{n=1}^{\infty}\frac{1}{a_na_{n+1}}.$$

Answer: $\frac{23-5\sqrt{21}}{10}$

Solution: The recurrence relation is given as:

$$a_{n+1}=\frac{5a_n+\sqrt{21a_n^2+4}}{2}.$$

We can rewrite this relation step by step as follows:

$$a_{n+1}=\frac{5a_n+\sqrt{21a_n^2+4}}{2}\implies 2a_{n+1}-5a_n=\sqrt{21a_n^2+4}$$

$$\implies 4a_{n+1}^2-20a_na_{n+1}+25a_n^2=21a_n^2+4\implies a_n^2-5a_na_{n+1}+a_{n+1}^2-1=0.$$

It follows that a_n is a solution to the quadratic equation: $r^2-5a_{n+1}r+a_{n+1}^2-1=0$. By symmetry, a_{n+2} is also a solution to: $r^2-5a_{n+1}r+a_{n+1}^2-1=0$.

Notice that $\{a_n\}$ is strictly increasing, so a_n, a_{n+2} are distinct roots of this quadratic. By Vieta's formulas, $a_n+a_{n+2}=5a_{n+1}$.

Now, we compute for $n\geq 2$:

$$\frac{1}{a_na_{n+1}}=\frac{a_n^2-5a_na_{n+1}+a_{n+1}^2}{a_na_{n+1}}=\frac{a_{n+1}}{a_n}+\frac{a_n}{a_{n+1}}-5=\frac{5a_n-a_{n-1}}{a_n}+\frac{a_n}{a_{n+1}}-5$$

$$=\frac{a_n}{a_{n+1}}-\frac{a_{n-1}}{a_n}$$

where we used $a_n^2-5a_na_{n+1}+a_{n+1}^2=1$ and $a_{n-1}+a_{n+1}=5a_n$. From $a_1=5$, we have $a_2=\frac{5a_1+\sqrt{21a_1^2+4}}{2}=24$. Now, the first N terms of the sum (called a partial sum) telescopes:

$$\sum_{n=1}^{N}\frac{1}{a_na_{n+1}}=\frac{1}{a_1a_2}+\sum_{n=2}^{N}\left(\frac{a_n}{a_{n+1}}-\frac{a_{n-1}}{a_n}\right)$$

$$=\frac{1}{120}+\frac{a_N}{a_{N+1}}-\frac{5}{24}=\frac{a_N}{a_{N+1}}-\frac{1}{5}.$$

The characteristic equation for $\{a_n\}$ (found by rewriting $a_{n+2}-5a_{n+1}+a_n=0$ as a quadratic) is:

$$r^2-5r+1=0.$$

Thus, solving for r with the quadratic formula, the general solution for a_n is:

$$a_n = c_1\left(\frac{5-\sqrt{21}}{2}\right)^n + c_2\left(\frac{5+\sqrt{21}}{2}\right)^n$$

where c_1 and c_2 are constants. As $N \to \infty$, we have:

$$\lim_{N\to\infty} \frac{a_N}{a_{N+1}} = \frac{1}{\frac{5+\sqrt{21}}{2}} = \frac{2(5-\sqrt{21})}{4} = \frac{5-\sqrt{21}}{2}$$

where we use the fact $c_1\left(\frac{5-\sqrt{21}}{2}\right)^n$ becomes really small for large n and c_2 cancels out. Thus, the final answer is:

$$\frac{5-\sqrt{21}}{2} - \frac{1}{5} = \boxed{\frac{23-5\sqrt{21}}{10}}.$$

10. Let $f(x) = 4x + a$, $g(x) = 6x + b$, and $h(x) = 9x + c$. Let $S(a, b, c)$ be the set of the 3^{20} functions formed by all possible compositions of f, g, h a total of 20 times, and let $R(a, b, c)$ be the number of distinct roots over all functions in $S(a, b, c)$. Compute the *second smallest* possible value of $R(a, b, c)$ as a, b, c range over all reals.

Answer: 41

Solution: Clearly, the smallest value is 1 which is obtained for $a = b = c = 0$ (here $S(a, b, c)$ only contains functions which are multiples of x). It is impossible to have no roots, since the composition of linear functions is linear. Now, let us suppose that $a \neq 0$.

Let us split into two cases:

1. All pairs of f, g, h commute under function composition (that is, $f \circ g = g \circ f$ and similarly for the other two pairs).
2. Some pair of two of f, g, h do not commute.

Let us first consider the first case. We must have that

$$f \circ g(x) = 24x + 4b + a = 24x + 6a + b = g \circ f(x) \Longrightarrow 5a = 3b$$

and similarly $8a = 3c$. This implies that all three of a, b, c are nonzero and that we can express the functions as $f(x) = 4x + \frac{4-1}{3}a$, $g(x) = 6x + \frac{6-1}{3}a$, $h(x) = 9x + \frac{9-1}{3}a$.

By induction, $f^\ell(x) = 4^\ell \cdot x + \frac{4^\ell - 1}{3} \cdot a$ (where $f^\ell = \underbrace{f \circ \ldots \circ f}_{\ell}$). Similarly, we find that

$$f^\ell \circ g^m \circ h^n(x) = 4^\ell \cdot 6^m \cdot 9^n \cdot x + \frac{4^\ell \cdot 6^m \cdot 9^n - 1}{3} \cdot a$$

so the root of this equation is

$$\frac{a}{3} \cdot \left(1 - \frac{1}{2^{2\ell+m} \cdot 3^{2n+m}}\right).$$

Therefore, the number of possible roots is equal to the number of possible values of $2\ell + m$ subject to $\ell + m + n = 20$ (since $2n + m$ is unique given the previous two statements). There are 41 possible values for this: 0, 1, ..., 40 which are all achievable. So, our first candidate for the second smallest value is 41.

Now, suppose that some pair do not commute: without loss of generality, suppose that $f \circ g \neq g \circ f$. Then, we can write $b = \frac{5}{3}a + d$ for some nonzero d.

Using this, we claim that $R(a, b, c) \geq \binom{20}{2} > 41$. Indeed, consider the subset of $S(a, b, c)$ generated by applying f 2 times and g 18 times.

Then, we can write any such application as $g^\ell \circ f \circ g^m \circ f \circ g^n$, where $\ell + m + n = 18$ (if they are 0, we treat this as applying the identity function). Let $b = \frac{5}{3}a + d$: then, we can directly compute this application as

$$\begin{aligned} g^\ell \circ f \circ g^m \circ f \circ g^n(x) &= 6^\ell\left(4\left(6^m\left(4\left(6^n \cdot x + \frac{6^n - 1}{3}a\right) + b\right) + \frac{6^m - 1}{3}a\right) + b\right) + \frac{6^\ell - 1}{3}a \\ &= 6^{18} \cdot 4^2 \cdot x + \frac{a}{3}\left(6^\ell - 1 + 6^\ell \cdot 4 \cdot (6^m - 1) + 6^{\ell+m} \cdot 4^2 \cdot (6^n - 1)\right) + b\left(6^\ell + 4 \cdot 6^{\ell+m}\right) \\ &= 6^{18} \cdot 4^2 \cdot x + d\left(6^\ell + 4 \cdot 6^{\ell+m}\right) + \frac{a}{3} \cdot (4^2 \cdot 6^{18} - 1). \end{aligned}$$

Now, since $d \neq 0$, we claim that every possible value of ℓ and m gives a different value for $6^\ell + 4 \cdot 6^{\ell+m}$ which implies the conclusion. Indeed, for any fixed value of $\ell + m$ it is clear that the values are all distinct. Else, note that $6^\ell + 4 \cdot 6^{\ell+m} < 6^{\ell+m+1}$. This implies that two different values of $\ell + m$ cannot collide.

We can repeat this same argument for the cases where the pairs f, h or g, h do not commute, and get the exact same outcome. Therefore, we are done and the answer is $\boxed{41}$.

TB. *This is an estimation question used for tiebreaking purposes. Ties on this test will be broken by absolute distance from the correct answer on this question.* Let

$$A = \sum_{k=0}^{5000^2-1} \frac{1}{k - 4999\left\lfloor \frac{k}{5000} \right\rfloor + 1}.$$

Estimate A in the decimal form $abcdef.ghij$ where $a, b, c, d, e, f, g, h, i, j$ are decimal digits each between 0 and 9, inclusive (leading zeros are possible).

Answer: 6930.9718

Solution: Let $n = 5000$. Note that

$$k - (n-1)\left\lfloor \frac{k}{n} \right\rfloor = \left\lfloor \frac{k}{n} \right\rfloor + \left(k - n\left\lfloor \frac{k}{n} \right\rfloor\right).$$

Let $i = \left\lfloor \frac{k}{n} \right\rfloor$ and $j = k - n\left\lfloor \frac{k}{n} \right\rfloor$. These are the quotient and remainder when n is divided by k, respectively. Thus, i and j are independent variables ranging from 0 to $n - 1$, and we can reformulate the sum as:

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{1}{i + j + 1}.$$

Let $s = i + j$. For $s \leq n - 1$, there are $s + 1$ possible pairs of (i, j) that sum to s, and for $s \geq n$, there are $2n - 1 - s$ possible pairs. Therefore:

$$\begin{aligned}
&\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{1}{i+j+1} \\
&= \sum_{s=0}^{n-1} \frac{s+1}{s+1} + \sum_{s=n}^{2n-2} \frac{2n-1-s}{s+1} \\
&= n + \left(\sum_{s=n}^{2n-2} \frac{2n}{s+1} \right) - (n-1) \\
&= 1 + 2n \left(\sum_{s=n}^{2n-2} \frac{1}{s+1} \right) \\
&= 1 + 2n(H_{2n-1} - H_n)
\end{aligned}$$

where H_k is the k-th harmonic number: $H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$.

We may estimate H_k as $\ln(k) + \gamma$ where γ is the Euler-Mascheroni constant.

$$H_{2n-1} - H_n \approx \ln(2n-1) - \ln(n) \approx \ln(2n) - \ln(n) = \ln 2.$$

Using this method, the approximated sum turns out to be

$$1 + 2n(\ln 2) \approx 1 + (10000)(0.693) = 6931.$$

The actual sum turns out to be: $\boxed{6930.9718305994529691723237145816594307 63}$.