2022 AIME I Problems

Problem 1

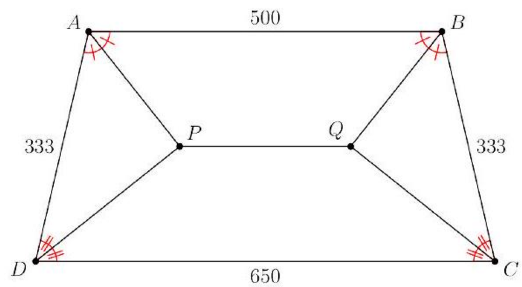
Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.

Problem 2

Find the three-digit positive integer $\underline{abc}$ whose representation in base nine is $\underline{bca}_{\text{nine}}$, where $a, b,$ and c are (not necessarily distinct) digits.

Problem 3

In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333$. The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.



Problem 4

Let $w = \frac{\sqrt{3}+i}{2}$ and $z = \frac{-1+i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

Problem 5

A straight river that is 264 meters wide flows from west to east at a rate of 14 meters per minute. Melanie and Sherry sit on the south bank of the river with Melanie a distance of D meters downstream from Sherry. Relative to the water, Melanie swims at 80 meters per minute, and Sherry swims at 60 meters per minute. At the same time, Melanie and Sherry begin swimming in straight lines to a point on the north bank of the river that is equidistant from their starting positions. The two women arrive at this point simultaneously. Find D.

Problem 6

Find the number of ordered pairs of integers (a, b) such that the sequence

$$3, 4, 5, a, b, 30, 40, 50$$

is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression.

Problem 7

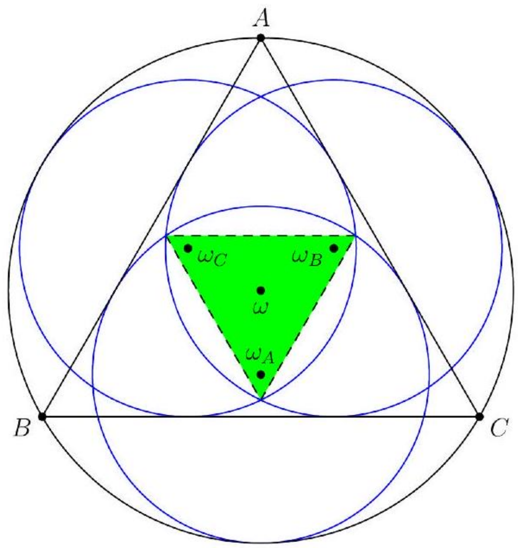
Let $a, b, c, d, e, f, g, h, i$ be distinct integers from 1 to 9. The minimum possible positive value of

$$\frac{a \cdot b \cdot c - d \cdot e \cdot f}{g \cdot h \cdot i}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 8

Equilateral triangle $\triangle ABC$ is inscribed in circle ω with radius 18. Circle ω_A is tangent to sides $\overline{AB}$ and $\overline{AC}$ and is internally tangent to ω. Circles ω_B and ω_C are defined analogously. Circles ω_A, ω_B, and ω_C meet in six points-two points for each pair of circles. The three intersection points closest to the vertices of $\triangle ABC$ are the vertices of a large equilateral triangle in the interior of $\triangle ABC$, and the other three intersection points are the vertices of a smaller equilateral triangle in the interior of $\triangle ABC$. The side length of the smaller equilateral triangle can be written as $\sqrt{a} - \sqrt{b}$, where a and b are positive integers. Find $a + b$.



Problem 9

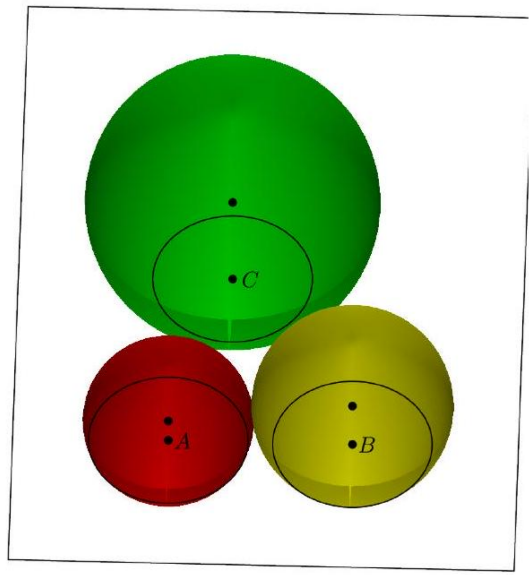
Ellina has twelve blocks, two each of red (**R**), blue (**B**), yellow (**Y**), green (**G**), orange (**O**), and purple (**P**). Call an arrangement of blocks even if there is an even number of blocks between each pair of blocks of the same color. For example, the arrangement

R B B Y G G Y R O P P O

is even. Ellina arranges her blocks in a row in random order. The probability that her arrangement is even is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

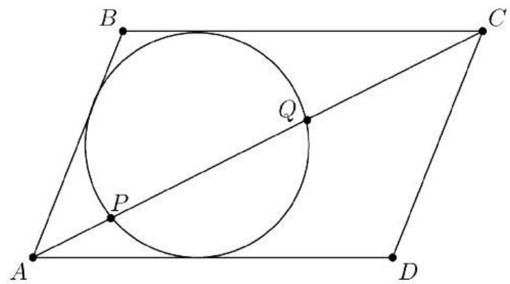
Problem 10

Three spheres with radii 11, 13, and 19 are mutually externally tangent. A plane intersects the spheres in three congruent circles centered at A, B, and C, respectively, and the centers of the spheres all lie on the same side of this plane. Suppose that $AB^2 = 560$. Find AC^2



Problem 11

Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. A circle tangent to sides $\overline{DA}, \overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3, PQ = 9$, and $QC = 16$. Then the area of $ABCD$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



Problem 12

For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \cdots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\}$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Problem 13

Let S be the set of all rational numbers that can be expressed as a repeating decimal in the form $0.\overline{abcd}$, where at least one of the digits $a, b, c,$ or d is nonzero. Let N be the number of distinct numerators obtained when numbers in S are written as fractions in lowest terms. For example, both 4 and 410 are counted among the distinct numerators for numbers in S because $0.\overline{3636} = \frac{4}{11}$ and $0.\overline{1230} = \frac{410}{3333}$. Find the remainder when N is divided by 1000.

Problem 14

Given $\triangle ABC$ and a point P on one of its sides, call line ℓ the splitting line of $\triangle ABC$ through P if ℓ passes through P and divides $\triangle ABC$ into two polygons of equal perimeter. Let $\triangle ABC$ be a triangle where $BC = 219$ and AB and AC are positive integers. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and suppose that the splitting lines of $\triangle ABC$ through M and N intersect at 30°. Find the perimeter of $\triangle ABC$.

Problem 15

Let x, y, and z be positive real numbers satisfying the system of equations:

$$\sqrt{2x - xy} + \sqrt{2y - xy} = 1$$
$$\sqrt{2y - yz} + \sqrt{2z - yz} = \sqrt{2}$$
$$\sqrt{2z - zx} + \sqrt{2x - zx} = \sqrt{3}$$

Then $[(1-x)(1-y)(1-z)]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.