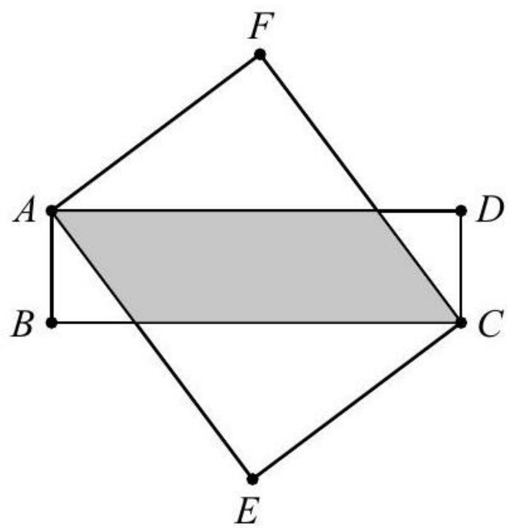
2021 AIME I Problems

Problem 1

Zou and Chou are practicing their 100-meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is $\frac{2}{3}$ if they won the previous race but only $\frac{1}{3}$ if they lost the previous race. The probability that Zou will win exactly 5 of the 6 races is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 2

In the diagram below, $ABCD$ is a rectangle with side lengths $AB = 3$ and $BC = 11$, and $AECF$ is a rectangle with side lengths $AF = 7$ and $FC = 9$. The area of the shaded region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 3

Find the number of positive integers less than 1000 that can be expressed as the difference of two integral powers of 2.

Problem 4

Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

Problem 5

Call a three-term strictly increasing arithmetic sequence of integers special if the sum of the squares of the three terms equals the product of the middle term and the square of the common difference. Find the sum of the third terms of all special sequences.

Problem 6

Segments $\overline{AB}, \overline{AC}$, and $\overline{AD}$ are edges of a cube and segment $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $BP = 60\sqrt{10}, CP = 60\sqrt{5}, DP = 120\sqrt{2}$, and $GP = 36\sqrt{7}$. Find AP.

Problem 7

Find the number of pairs (m, n) of positive integers with $1 \leq m < n \leq 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2$$

Problem 8

Find the number of integers c such that the equation

$$||20|x| - x^2| - c| = 21$$

has 12 distinct real solutions.

Problem 9

Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from A to the lines BC, CD, and BD are $15, 18$, and 10, respectively. Let K be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

Problem 10

Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m+18}{n+19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

Problem 11

Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 12

Let $A_1A_2A_3\dots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 13

Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB intersects ω at two points P and Q such that the measure of minor arc PQ is 120°. Find the distance between the centers of ω_1 and ω_2.

Problem 14

For any positive integer $a, \sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma\left(a^n\right) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

Problem 15

Let S be the set of positive integers k such that the two parabolas

$$y = x^2 - k \quad \text{and} \quad x = 2(y-20)^2 - k$$

intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.