2017 AIME I Problems

Problem 1

Fifteen distinct points are designated on $\triangle ABC:$ the 3 vertices $A, B,$ and C; 3 other points on side $\overline{AB}$; 4 other points on side $\overline{BC}$; and 5 other points on side $\overline{CA}$. Find the number of triangles with positive area whose vertices are among these 15 points.

Problem 2

When each of $702, 787$, and 855 is divided by the positive integer m, the remainder is always the positive integer r. When each of $412, 722$, and 815 is divided by the positive integer n, the remainder is always the positive integer $s \neq r$. Find $m + n + r + s$.

Problem 3

For a positive integer n, let d_n be the units digit of $1+2+3+\cdots+n$. Find the remainder when

$$\sum_{n=1}^{2017} d_n$$

is divided by 1000.

Problem 4

A pyramid has a triangular base with side lengths $20, 20$, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Problem 5

A rational number written in base eight is $\underline{ab}.\underline{cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb}.\underline{ba}$. Find the base-ten number $\underline{abc}$.

Problem 6

A circle is circumscribed around an isosceles triangle whose two congruent angles have degree measure x. Two points are chosen independently and uniformly at random on the circle, and a chord is drawn between them. The probability that the chord intersects the triangle is $\frac{14}{25}$. Find the difference between the largest and smallest possible values of x.

Problem 7

For nonnegative integers a and b with $a+b \leq 6$, let $T(a,b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a,b)$, where a and b are nonnegative integers with $a+b \leq 6$. Find the remainder when S is divided by $1000.$.

Problem 8

Two real numbers a and b are chosen independently and uniformly at random from the interval $(0, 75)$. Let O and P be two points in the plane with $OP = 200$. Let Q and R be points on the same side of line OP such that the degree measures of $\angle POQ$ and $\angle POR$ are a and b, respectively, and $\angle OQP$ and $\angle ORP$ are both right angles. The probability that $QR \leq 100$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 9

Let $a_{10} = 10$, and for each integer $n > 10$ let $a_n = 100a_{n-1} + n$. Find the least $n > 10$ such that a_n is a multiple of 99.

Problem 10

Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

Problem 11

Consider arrangements of the 9 numbers $1, 2, 3, \ldots, 9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows $1, 2$, and 3, respectively, and then let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

Problem 12

Call a set S product-free if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Problem 13

For every $m \geq 2$, let $Q(m)$ be the least positive integer with the following property: For every $n \geq Q(m)$, there is always a perfect cube k^3 in the range $n < k^3 \leq m \cdot n$. Find the remainder when

$$\sum_{m=2}^{2017} Q(m)$$

is divided by 1000.

Problem 14

Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

Problem 15

The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}, 5$, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.