2023 AIME II Problems

Problem 1

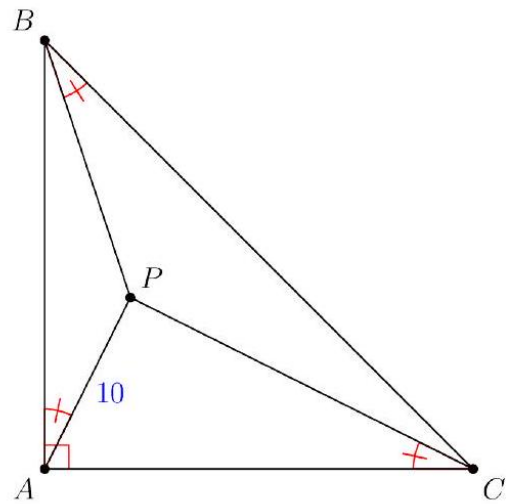
The numbers of apples growing on each of six apple trees form an arithmetic sequence where the greatest number of apples growing on any of the six trees is double the least number of apples growing on any of the six trees. The total number of apples growing on all six trees is 990. Find the greatest number of apples growing on any of the six trees.

Problem 2

Recall that a palindrome is a number that reads the same forward and backward. Find the greatest integer less than 1000 that is a palindrome both when written in base ten and when written in base eight, such as $292 = 444_{\text{eight}}$.

Problem 3

Let $\triangle ABC$ be an isosceles triangle with $\angle A = 90^\circ$. There exists a point P inside $\triangle ABC$ such that $\angle PAB = \angle PBC = \angle PCA$ and $AP = 10$. Find the area of $\triangle ABC$.



Problem 4

Let x, y, and z be real numbers satisfying the system of equations

$$xy + 4z = 60$$
$$yz + 4x = 60$$
$$zx + 4y = 60$$

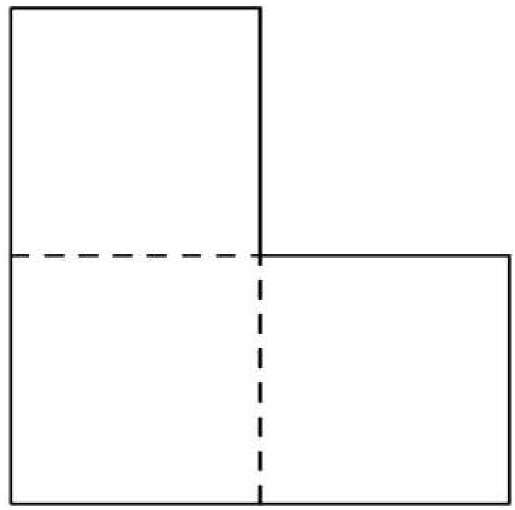
Let S be the set of possible values of x. Find the sum of the squares of the elements of S.

Problem 5

Let S be the set of all positive rational numbers r such that when the two numbers r and $55r$ are written as fractions in lowest terms, the sum of the numerator and denominator of one fraction is the same as the sum of the numerator and denominator of the other fraction. The sum of all the elements of S can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 6

Consider the L-shaped region formed by three unit squares joined at their sides, as shown below. Two points A and B are chosen independently and uniformly at random from inside the region. The probability that the midpoint of $\overline{AB}$ also lies inside this L-shaped region can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 7

Each vertex of a regular dodecagon (12-gon) is to be colored either red or blue, and thus there are 2^{12} possible colorings. Find the number of these colorings with the property that no four vertices colored the same color are the four vertices of a rectangle.

Problem 8

Let $\omega = \cos \frac{2\pi}{7} + i \cdot \sin \frac{2\pi}{7}$, where $i = \sqrt{-1}$. Find the value of the product

$$\prod_{k=0}^{6} \left(\omega^{3k} + \omega^{k} + 1\right)$$

Problem 9

Circles ω_1 and ω_2 intersect at two points P and Q, and their common tangent line closer to P intersects ω_1 and ω_2 at points A and B, respectively. The line parallel to AB that passes through P intersects ω_1 and ω_2 for the second time at points X and Y, respectively. Suppose $PX = 10, PY = 14$, and $PQ = 5$. Then the area of trapezoid $XABY$ is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Problem 10

Let N be the number of ways to place the integers 1 through 12 in the 12 cells of a 2×6 grid so that for any two cells sharing a side, the difference between the numbers in those cells is not divisible by 3. One way to do this is shown below. Find the number of positive integer divisors of N.

1	3	5	7	9	11
2	4	6	8	10	12

Problem 11

Find the number of collections of 16 distinct subsets of $\{1, 2, 3, 4, 5\}$ with the property that for any two subsets X and Y in the collection, $X \cap Y \neq \emptyset$.

Problem 12

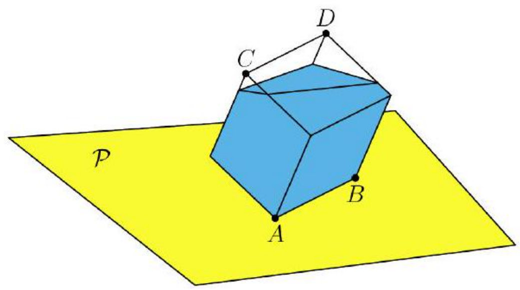
In $\triangle ABC$ with side lengths $AB = 13, BC = 14$, and $CA = 15$, let M be the midpoint of $\overline{BC}$. Let P be the point on the circumcircle of $\triangle ABC$ such that M is on $\overline{AP}$. There exists a unique point Q on segment $\overline{AM}$ such that $\angle PBQ = \angle PCQ$. Then AQ can be written as $\frac{m}{\sqrt{n}}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 13

Let A be an acute angle such that $\tan A = 2\cos A$. Find the number of positive integers n less than or equal to 1000 such that $\sec^n A + \tan^n A$ is a positive integer whose units digit is 9.

Problem 14

A cube-shaped container has vertices A, B, C, and D, where $\overline{AB}$ and $\overline{CD}$ are parallel edges of the cube, and $\overline{AC}$ and $\overline{BD}$ are diagonals of faces of the cube, as shown. Vertex A of the cube is set on a horizontal plane $\mathcal{P}$ so that the plane of the rectangle $ABDC$ is perpendicular to $\mathcal{P}$, vertex B is 2 meters above $\mathcal{P}$, vertex C is 8 meters above $\mathcal{P}$, and vertex D is 10 meters above $\mathcal{P}$. The cube contains water whose surface is parallel to $\mathcal{P}$ at a height of 7 meters above $\mathcal{P}$. The volume of water is $\frac{m}{n}$ cubic meters, where m and n are relatively prime positive intgers. Find $m + n$.



Problem 15

For each positive integer n let a_n be the least positive integer multiple of 23 such that $a_n \equiv 1 \pmod{2^n}$. Find the number of positive integers n less than or equal to 1000 that satisfy $a_n = a_{n+1}$.