2018 AIME II Problems

Problem 1

Points A, B, and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

Problem 2

Let $a_0 = 2, a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4\left(a_{n-1} + a_{n-2} + a_{n-3}\right)$ is divided by 11 . Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

Problem 3

Find the sum of all positive integers $b < 1000$ such that the base- b integer 36_b is a perfect square and the base- b integer 27_b is a perfect cube.

Problem 4

In equiangular octagon CAROLINE, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon CORNELIA encloses six non-overlapping triangular regions. Let K be the area enclosed by CORNELIA, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Problem 5

Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.

Problem 6

A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 + (-4a+3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 7

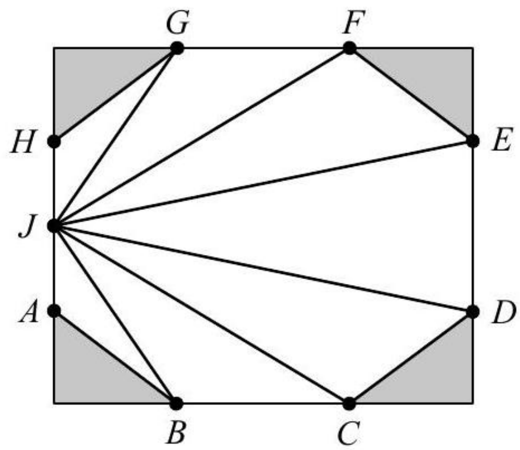
Triangle ABC has side lengths $AB = 9, BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0, P_1, P_2, \ldots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \ldots, 2449$, and points $A = Q_0, Q_1, Q_2, \ldots, Q_{2450} = C$ are on segment $\overline{AC}$ with Q_k between Q_{k-1} and Q_{k+1} for $k = 1, 2, \ldots, 2449$. Furthermore, each segment $\overline{P_kQ_k}, k = 1, 2, \ldots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions has the same area. Find the number of segments $\overline{P_kQ_k}, k = 1, 2, \ldots, 2450$, that have rational length.

Problem 8

A frog is positioned at the origin in the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x+1, y), (x+2, y), (x, y+1)$, or $(x, y+2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

Problem 9

Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four $6-8-10$ triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{HA}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}, \overline{JC}, \overline{JD}, \overline{JE}, \overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



Problem 10

Find the number of functions $f(x)$ from $\{1,2,3,4,5\}$ to $\{1,2,3,4,5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1,2,3,4,5\}$.

Problem 11

Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.

Problem 12

Let $ABCD$ be a convex quadrilateral with $AB = CD = 10, BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

Problem 13

Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 14

The incircle ω of $\triangle ABC$ is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ and ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3, PB = 4, AC = 8,$ and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 15

Find the number of functions f from $\{0,1,2,3,4,5,6\}$ to the integers such that $f(0)=0$, $f(6)=12$, and

$$|x-y| \leq |f(x)-f(y)| \leq 3|x-y|$$

for all x and y in $\{0,1,2,3,4,5,6\}$.